MATH 105A and 110A Review: The determinant and invertibility

1. Find the inverse of the Jacobian matrix of $F(x,y) = (2xy, x+y)$ when possible.

Solution: Sine $F : \mathbb{R}^2 \to \mathbb{R}^2$, then the Jacobian matrix of F is a 2×2 matrix. We have

$$\nabla F = \begin{bmatrix} 2y & 2x \\ 1 & 1 \end{bmatrix}.$$

∇F is invertible if and only if $0 \neq \nabla F$. We have $0 = \nabla F = 2y - 2x$ if and only if $x = y$. Hence, the matrix is invertible if and only if $x \neq y$ and the inverse is

$$(\nabla F)^{-1} = \frac{1}{2y - 2x} \begin{bmatrix} 2y & 2x \\ 1 & 1 \end{bmatrix}.$$

2. Find the inverse of the Jacobian matrix $F(x,y,z) = (z, yz+2, 3xyz)$ at the point $(1,1,1)$.

Solution: The Jacobian of F is

$$\nabla F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & z & y \\ 3yz & 3xz & 3xy \end{bmatrix}.$$

At the point $(1,1,1)$, we have

$$\nabla F(1,1,1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} 1/3R_1 \to R_1 \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1/3 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1/3 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} R_1 - R_2 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1/3 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} R_2 - R_3 \to R_2 \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1/3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Thus,

$$(\nabla F(1,1,1))^{-1} = \begin{bmatrix} 0 & -1 & 1/3 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$